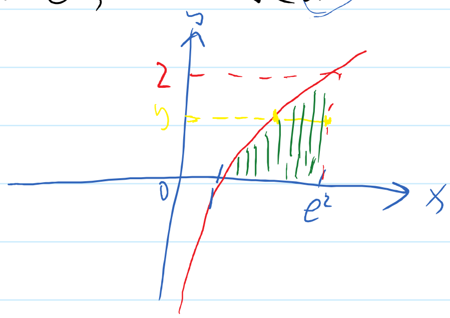
2020年9月23日 10:12

MATH 2020 Tutorial 2

① Sketch the described regions of integration.

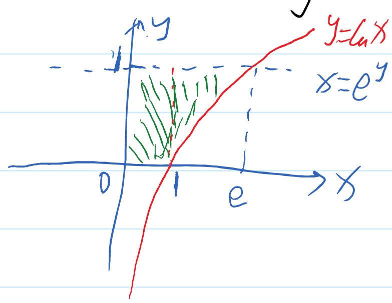
$1 \le x \le e^2, \quad 0 \le y \le \ln x$



$$\int_1^{e^2}\int_0^{\ln x} dy\,dx$$

$$= \int_0^2 \int_{e^y}^{e^2} dx\,dy$$

② The region bounded by $y=0$, $x=0$, $y=1$, $y=\ln x$



$$\int_0^1\int_0^1 dy\,dx + \int_1^e\int_{\ln x}^1 dy\,dx$$

$$= \int_0^1\int_0^{e^y} dx\,dy$$

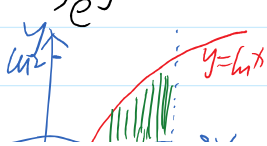
③ $\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\,dx\,dy$

$$= \int_0^1 3y^2 e^{xy}\Big|_0^{y^2} dy$$

$$= \int_0^1 3y^2 e^{y^3} - 3y^2\,dy$$

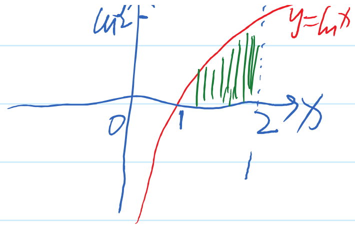
$$= e^{y^3} - y^3\Big|_0^1 = (e-1)-(1) = e-2$$

④ $\int_0^{\ln 2}\int_{e^y}^2 dx\,dy \quad = \int_1^2\int_0^{\ln x} dy\,dx$



$x=e^y$

$y=\ln x$



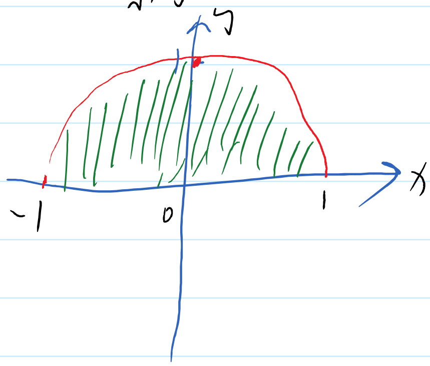
$x = e^y$

$y = \ln x$

⑤ $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\,dy = \int_{-1}^{1} \int_0^{\sqrt{1-x^2}} dy\,dx$

$x = \sqrt{1-y^2}$

$x^2+y^2=1$



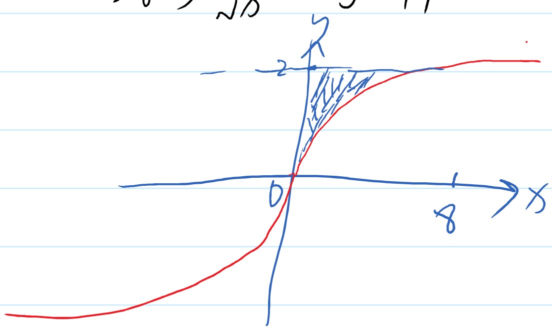
⑥ $\int_0^8 \int_{\sqrt[3]{x}}^{2} \frac{1}{y^4+1}\,dy\,dx = \int_0^2 \int_0^{y^3} \frac{1}{y^4+1}\,dx\,dy$ [Fubini's Thm]

$= \int_0^2 \frac{x}{y^4+1} \Big|_0^{y^3} dy$

$= \int_0^2 \frac{y^3}{y^4+1}\,dy$

$= \frac{1}{4} \ln(y^4+1) \Big|_0^2$

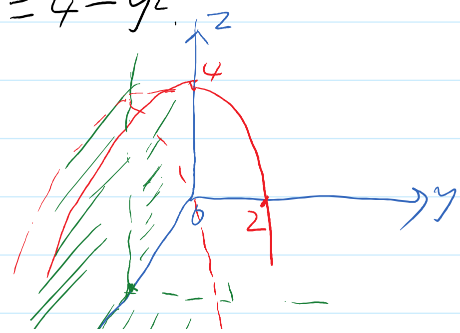
$= \frac{1}{4} \ln 17$



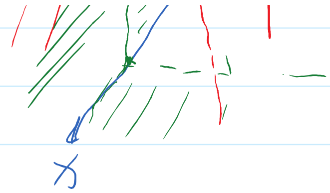
⑦ Find the volumn of solid in the first octant bounded by the coordinate planes, the plane $x=3$, and $z=4-y^2$.

$\int_0^3 \int_0^2 (4-y^2)\,dy\,dx$

$= \int_0^3 4y - \frac{1}{3}y^3 \Big|_0^2 dx$



$$= \int_0^3 4y - \frac{1}{3}y^3 \Big|_0 \, dx$$



$$= \int_0^3 \left(8 - \frac{8}{3}\right) dx$$

$$= \left(8 - \frac{8}{3}\right) \times (3-0) = 24 - 8 = 16$$

(8) Improper double integral

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(x^2+1)(y^2+1)} \, dx\, dy$$

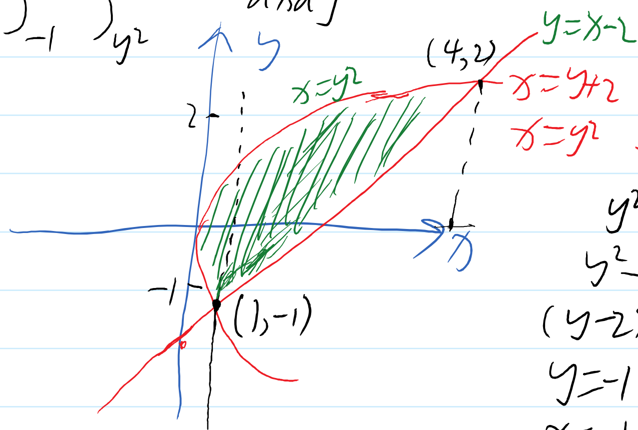
$$= 4 \int_0^{\infty} \boxed{\int_0^{\infty} \frac{1}{(x^2+1)(y^2+1)} \, dx}\, dy$$

$$= 4 \int_0^{\infty} \frac{1}{y^2+1} \lim_{b \to \infty} (\arctan b - \arctan 0) \, dy$$

$$= 4 \int_0^{\infty} \frac{1}{y^2+1} \, \frac{\pi}{2} \, dy$$

$$= 4 \cdot \frac{\pi}{2} \cdot \frac{\pi}{2} = \pi^2$$

(9) $$\int_{-1}^{2} \int_{y^2}^{y+2} dx\, dy$$



$\left.\begin{matrix} x = y+2 \\ x = y^2 \end{matrix}\right\} \Rightarrow \begin{cases} y = x-2 \\ y = \pm\sqrt{x} \end{cases}$

$y^2 = y+2$

$y^2 - y - 2 = 0$

$(y-2)(y+1) = 0$

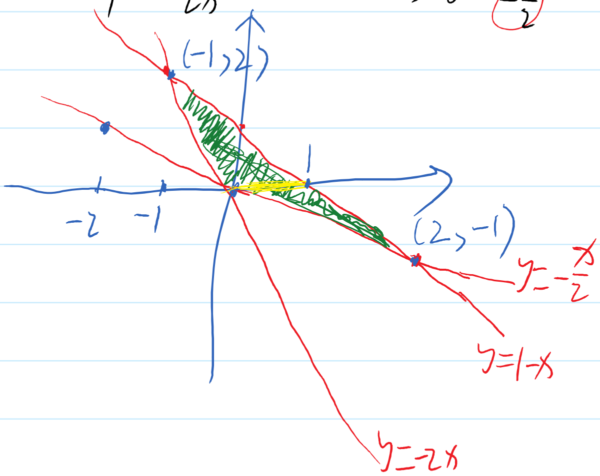
$y = -1$ or $y = 2$

$x = 1$ or $x = 4$

$$\int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} dy\, dx + \int_1^4 \int_{x-2}^{\sqrt{x}} dy\, dx$$

(10) $$\int^{0} \int^{1-x} dy\, dx + \int^{2} \int_{x}^{1-x} dy\, dx$$

(10) $\int_{-1}^{0}\int_{-2x}^{1-x} dydx + \int_{0}^{2}\int_{-\frac{x}{2}}^{1-x} dydx$



$y=1-x$

$y=-2x$

$y=-\frac{x}{2}$

$1-x=-\frac{x}{2}$

$\frac{x}{2}=1$

$x=2$

$1-x=-2x$

$-x=1$

$x=-1$

$\int_{-1}^{0}\int_{-2y}^{1-y} dxdy + \int_{0}^{2}\int_{-\frac{y}{2}}^{1-y} dxdy$

(11) Q2 in Supplementary Problems

if $f\geq 0$, then $\int_D f \geq 0$

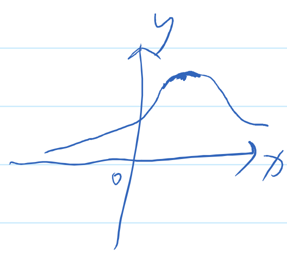
$A\leq f\leq B$ $\qquad B-f\geq 0$

$\int_D (B-f)\geq 0$

$B\int_D \chi_D - \int_D f\geq 0$

$\int_D f \leq B\cdot|D|$

Q3:

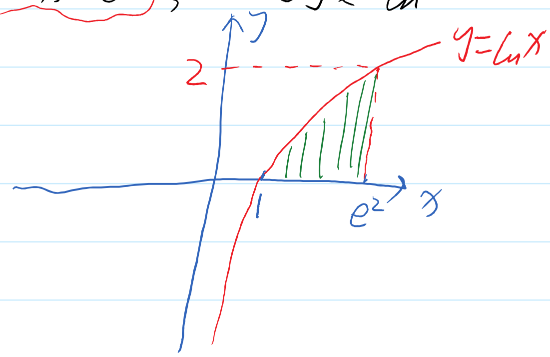


MATH 2020 Tutorial 2

(1) Sketch the described region of integration

① Sketch the described region of integration.

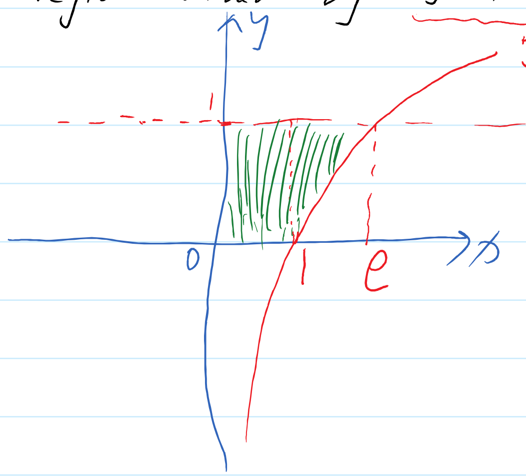
$1 \le x \le e^2$, $0 \le y \le \ln x$



$$\int_1^{e^2} \int_0^{\ln x} dy\,dx$$

$$= \int_0^2 \int_{e^y}^{e^2} dx\,dy$$

② The region bounded by $y=0$, $x=0$, $y=1$, $y = \ln x$



$y = \ln x \Rightarrow x = e^y$

$$\int_0^1 \int_0^1 dy\,dx + \int_1^e \int_{\ln x}^1 dy\,dx$$

$$= \int_0^1 \int_0^{e^y} dx\,dy$$

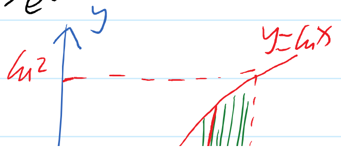
③ $$\int_0^1 \int_0^{y^2} 3y^3 e^{xy}\,dx\,dy$$

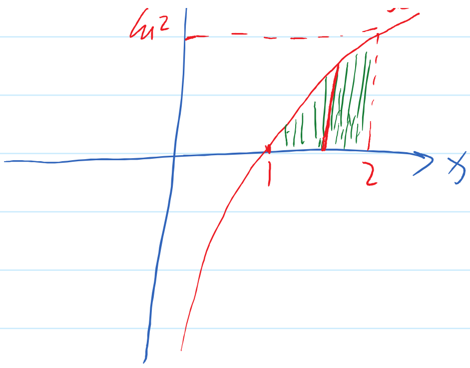
$$= \int_0^1 3y^2 e^{xy} \Big|_0^{y^2} dy$$

$$= \int_0^1 3y^2 e^{y^3} - 3y^2\,dy$$

$$= e^{y^3} - y^3 \Big|_0^1 = (e-1) - (1-0) = e-2$$

④ $$\int_0^{\ln 2} \int_{e^y}^2 dx\,dy \quad = \int_1^2 \int_0^{\ln x} dy\,dx$$

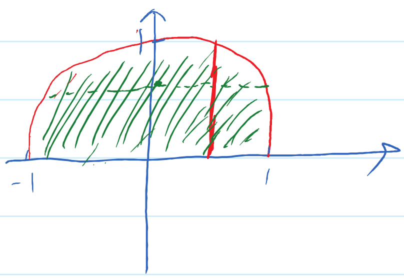




⑤ $$\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\,dy = \int_{-1}^{1} \int_0^{\sqrt{1-x^2}} dy\,dx$$

$x=\sqrt{1-y^2}$

$x^2+y^2=1$



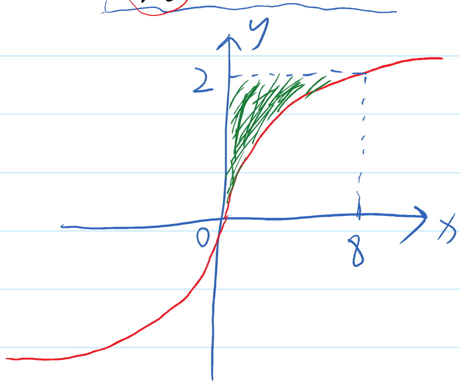
⑥ $$\int_0^8 \int_{\sqrt[3]{x}}^{2} \frac{1}{y^4+1}\,dy\,dx = \int_0^2 \int_0^{y^3} \frac{1}{y^4+1}\,dx\,dy$$ [Fubini's Thm]

$y=\sqrt[3]{x}$

$$= \int_0^2 \left.\frac{x}{y^4+1}\right|_0^{y^3} dy$$

$$= \int_0^2 \frac{y^3}{y^4+1}\,dy$$

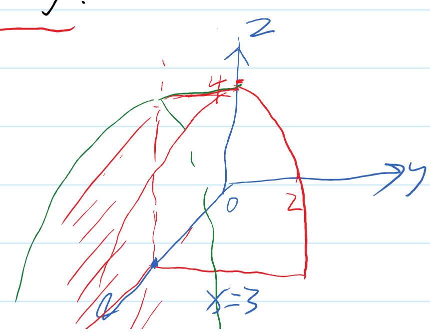
$$= \frac{1}{4}\cdot \ln(y^4+1)\Big|_0^2 = \frac{1}{4}\ln 17$$



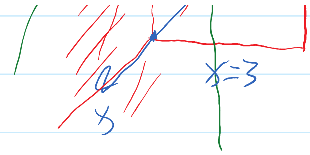
⑦ Find the volumn of solid in the first octant bounded by the coordinate planes, the plane $x=3$, and $z=4-y^2$.

$$\text{Volumn} = \int_0^3 \int_0^2 (4-y^2)\,dy\,dx$$

$$= \int_0^3 \left. 4y - \frac{1}{3}y^3 \right|_0^2 dx$$



$$= \int_0 4y - \frac{}{3}y^{} \Big|_0 \, dx$$



$$= \int_0^3 \left(8 - \frac{8}{3}\right) dx$$

$$= \left(8 - \frac{8}{3}\right)(3-0) = 24 - 8 = 16$$

⑧ Improper double integral.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(y^2+1)}\, dx\, dy$$

by definition $\lim\limits_{\substack{a,c\to\infty \\ b,d\to-\infty}} \int_b^a \int_d^c \frac{1}{(x^2+1)(y^2+1)}\, dx\, d$

$= \pi^2$.

$$= 4 \int_0^{\infty} \boxed{\int_0^{\infty} \frac{1}{(x^2+1)(y^2+1)}\, dx}\, dy$$

$$= 4 \int_0^{\infty} \frac{1}{y^2+1} \lim_{b\to\infty} (\arctan b - \arctan 0)\, dy$$

$$= 4 \int_0^{\infty} \frac{1}{y^2+1} \cdot \frac{\pi}{2}\, dy$$

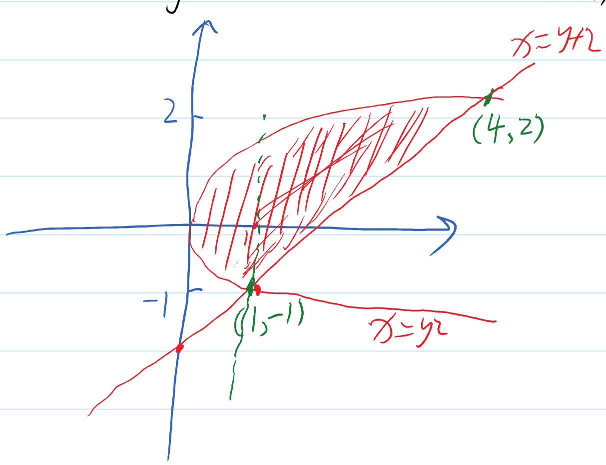
$$= 4 \cdot \frac{\pi}{2} \cdot \frac{\pi}{2} = \pi^2$$

$$\int_a^{\infty} f := \lim_{b\to\infty} \int_a^b f$$

$$\int_{-\infty}^{\infty} f := \lim_{a\to\infty} \lim_{b\to\infty} \int_a^b f$$

$$\int_a^b f := \lim_{c\to a} \lim_{d\to b} \int_c^d f$$

⑨ $$\int_{-1}^{2}\int_{y^2}^{y+2} dx\, dy = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} dy\, dx + \int_1^4 \int_{x-2}^{\sqrt{x}} dy\, dx$$



$$\begin{cases} x = y+2 \\ x = y^2 \end{cases} \Rightarrow y^2 = y+2$$

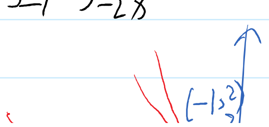
$$y^2 - y - 2 = 0$$

$$(y-2)(y+1) = 0$$

$y = -1$ or $y = 2$

$x = 1$ or $x = 4$

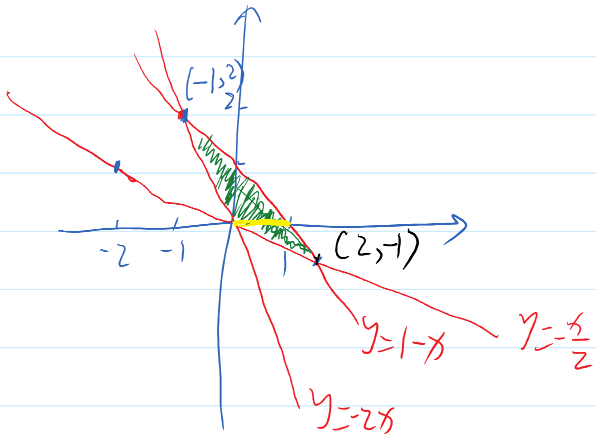
⑩ $$\int_{-1}^{0}\int_{-2x}^{1-x} dy\, dx + \int_0^2 \int_{-\frac{x}{2}}^{1-x} dy\, dx = \int_{-1}^{0}\int_{-2y}^{1-y} dx\, dy + \int_0^2 \int_{-\frac{y}{2}}^{1-y} dx\, dy$$



$y = 1-x$

$y = -2x$

$y = -\frac{x}{2}$



$y=1-x$

$y=-2x$

$y=-\frac{x}{2}$

2

$$\begin{cases} y=-\frac{x}{2} \\ y=1-x \end{cases} \Rightarrow 1-x=-\frac{x}{2}$$

$$\frac{x}{2}=1$$

$$x=2$$